Calculation Carrying Programs
– How to Code Program Transformations –

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Abstract

In this paper, we propose a new mechanism called calculation carrying programs *that can relax the tension between efficiency and clarity in programming. The idea is to accompany clear programs with some calculation specifying the intention of how to manipulate the programs to be efficient. This calculation specification can be executed automatically by our compiler to derive efficient programs. As a result, each calculation carrying program makes itself be a complete document including a concise specification of given problem as well as an effective way to derive both efficient and correct code.*

1 Introduction

Consider to write a program to check whether a list of numbers is *steep*. A list is said to be steep if each element of the list is greater than the average of the elements that follow it: a similar problem was discussed in [12]. A straightforward program to solve the problem is

$$
\begin{array}{lcl}
steep & :: & [Int] \to Bool \\
steep\ [\,] & = & True \\
steep\ (a : x) & = & (a > average\ x)\ \wedge\ steep\ x \\
 & & \\
average & :: & [Int] \to Int \\
average\ x & = & sum\ x\ /\ length\ x.
\end{array}
$$

This program, though being clear, is inefficient (a quadratic algorithm) due to repeated applications of *average* to the sublists. In fact, an efficient linear program does exist.

$$
\begin{array}{lcl}
steepOpt & :: & [Int] \to Bool \\
steepOpt\ x & = & \textbf{let}\ (st, s, l) = steep'\ x \\
 & & \textbf{in}\ st \\
steep'\ [\,] & = & (True, 0, 0) \\
steep'\ (a : x) & = & \textbf{let}\ (st, s, l) = steep'\ x \\
 & & \textbf{in}\ ((a > (s/l)) \wedge st,\ a + s,\ l + 1)
\end{array}
$$

Programmers are now forced to select one from the two programs by most practical programming systems, but this selection is essentially difficult.

- The straightforward one is of high readability and good modularity. It, however, comes at the cost of inefficiency, which may probably be intolerable. One may hope that a language compiler could automatically improve the program, but this is practically difficult. As far as we know, no popular Haskell compilers can automatically generate linear code from the straightforward program of *steep*.

- The efficient one is rather appealing, but it is far from being obvious why the program does correctly solve the problem without enough comment. Unfortunately, comment to the program is usually several lines in practice, which is too informal and too simple for program readers to understand algorithm completely. This makes the program difficult to be maintained, and even harder to be adapted to solve similar problems.

To remedy this situation, we shall propose a new mechanism called *calculation carrying programs* that can relax the tension between clarity and efficiency in programming. The idea is to accompany straightforward programs with some calculation specifying the intention of how to manipulate programs to be efficient. Thus, a calculation carrying program is not just means

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to show how to solve a problem, but also to show how to achieve improvement.

Program calculation is a kind of program transformation based on the theory of *Constructive Algorithmics* (also known as *Bird-Meertens Formalisms*) [4, 27, 14, 2], which is a program calculus for program derivation. Calculation is a series of applications of calculational laws (i.e. rules) that describe some properties of programs. Different from the popular *fold/unfold* program transformation technique [7], program calculation completely avoid using expensive *fold* steps at all.

We believe that it is both worthwhile and challenging to provide a flexible mechanism to code program calculations, and to make such calculations be part of programs. There are two main reasons. First, coding calculation is useful to help programmers to document and reuse their program development process. As argued in [12], a typical functional programmer usually develops his program using calculation on the back of an envelope, and only records the final result in his codes, like the above program for solving the steep problem. Of course, he could document his ideas in comments, but as we all know, this is rarely done. Furthermore, when the programmer finds himself in a similar situation using the same technique to develop a new piece of code, there is no way he can reuse the development recorded as a comment.

Second, coding calculation can help to mechanize derivation of efficient programs. Many calculation laws and theorems such as fusion [32], tupling [18], and parallelization [20] have been developed, but few of them have been fully implemented in practical compilers. There are two major difficulties. First, even for a simple calculation law like the cheap fusion in [15, 32], one cannot code it as *naturally* as expressed in the paper. Rather one has to take pain to *design* an algorithm to implement the law by induction on the syntax tree. Second, the *creative steps*, which are often required during calculation, are hard to be mechanized in general. By coding calculation, we can program these creative steps by ourselves and only leave those parts that can be mechanized for compiler.

This paper makes the first attempt at the design and implementation of a (functional) language that can support calculation carrying programs. In this language, programmers can write both a straightforward solution to a problem as they usually do, and a calculation declaring their intention for transforming the solution better. As a result, each calculation carrying program makes itself be a complete document including a concise specification of given problem as well as an effective way to derive both efficient and correct code.

We have implemented an experimental programming environment to support developing calculation carrying programs [30]. The main point is that the system can record program development and automatically generate a calculation carrying program. Therefore, with the system one can reuse the development to solve similar problems.

The rest of this paper is organized as follows. We illustrate informally the basic idea of calculation carrying programs by two simple examples in Section 2. Then in Section 3, we give the formal definition of the core language for writing the calculation carrying programs, including its syntax, semantics and typing rules. More application examples for coding calculation rules, calculation strategies, and development processes are given in Section 4. Finally, we discuss the related work and make a concluding remarks in Section 5 and Section 6 respectively.

2 An Overview

Each calculation carrying program consists of two parts: an object program that describes concise solution to given problem, and a meta program that describes how to improve the object program to be more efficient one. In this section, we will illustrate informally the basic idea by two simple examples. More practical examples can be found in Section 4.

2.1 Coding Rules using Matching

Before giving a whole calculation carrying program, we start by demonstrate how to code the following famous *fusion* calculation rule:

$$\frac{\begin{array}{rcl} f\ e & = & e' \\ f\ (a \oplus r) & = & a \otimes f\ r \end{array}}{f \circ foldr\ (\oplus)\ e = foldr\ (\otimes)\ e'}$$

reading that one can fuse the composition of a function f and a $foldr$ structure into a single $foldr$, provided that the two promotable conditions are satisfied. This rule plays an important role not only in calculating efficient functional programs [5, 27], but also in compiler construction like the warm fusion [25] in the Glasgow Haskell Compiler (GHC).

Even for such a simple calculation rule just in three lines, it requires a tedious work to implement it. The reason is that one cannot code the rule as *naturally* as expressed as above. Rather one has to *reinvent* an algorithm to implement the law by induction on the syntax tree of programs. To remedy this situation, we introduce explicit *matching* to our language.

Recall that pattern matching is a well-appreciated concept of functional programming [28]. It contributes to concise function definitions by implicitly decomposing data type values. For example, the following defines *length*, a function to compute the length of a list, according to two list patterns; an empty list [], and a list whose head element is a and the rest list is x.

$$\begin{array}{lcl} \mathit{length}\ [\,] & = & 0 \\ \mathit{length}\ (a : x) & = & 1 + \mathit{length}\ x \end{array}$$

Pattern matching indeed provides a good way for describing manipulation of data, but it does not fulfill our needs to specify manipulation of programs. If we would insist on using usual pattern matching of program syntax trees to code the fusion calculation rule, we would need a second invention to express the procedure showing how to derive $\otimes$ from $\oplus$ and f by traversing the syntax trees explicitly, which would lead to a rather long program.

Our idea is to relieve pattern matching from the restriction that pattern to be matched must be constructed by data constructors and pattern variables, allowing any *expression* (higher order patterns) to be a target for matching. With this general matching, we can now code fusion simply as follows.

$$\begin{array}{l} \mathit{fusion}\ ::\ <[a] \to b> \ \to\ <[a] \to b> \\ \mathit{fusion}\ <f \circ \mathit{foldr}\ (\oplus)\ e> \\ \quad = \ \mathbf{letrn} \\ \qquad\qquad e' \quad = \quad f\ e \\ \qquad\qquad a \otimes r \quad = \quad f\ (a \oplus x) \quad \Leftarrow \ r == f\ x \\ \quad \mathbf{in} \\ \qquad\qquad <\mathit{foldr}\ (\otimes)\ e'> \end{array}$$

The *fusion* defines a calculation transforming the code of an expression with type $[a] \to b$ to another. We use brackets <> to surround expressions (or types) to denote expression codes (or code types). To see the meaning of the definition of *fusion*, we demonstrate how

$$\mathit{fusion}\ <\mathit{length} \circ \mathit{foldr}\ (:)\ [\,]>$$

works. Upon receiving the expression code of $<\mathit{length} \circ \mathit{foldr}\ (:)\ [\,]>$, *fusion* matches it with $<f \circ \mathit{foldr}\ (\oplus)\ e>$ to bind f, $\oplus$ and e, and gets

$$\begin{array}{lcl} f & \mapsto & \mathit{length} \\ \oplus & \mapsto & : \\ e & \mapsto & [\,]. \end{array}$$

Then with these bindings, we reduce $f\ e$ to a form that does not contain any β and η redex, and then match it with e' to bind e', and gets

$$e' \ \mapsto \ 0$$

since the reduction of *length* [] gives 0. Similarly, matching the reduction result of $f\ (a \oplus x)$ with $a \otimes r$ gives several bindings for $\otimes$ and r, from which we only choose one such that r is syntactically equivalent to *length* x:

$$\otimes \ \mapsto \ \lambda a.\lambda x.\,(1 + x)$$

Finally, we build the code by replacing $\otimes$ and e' by their bindings in $<\mathit{foldr}\ (\otimes)\ e'>$, and get

$$<\mathit{foldr}\ (\lambda a.\lambda x.\,(1 + x))\ 0>\,.$$

Formal account of the meaning of the language can be found in Section 3.

2.2 A Calculation Carrying Program

Figure 1 gives the example of a complete calculation carrying program for the steep problem. It contains three parts: a straightforward program, a calculation, and a relationship between them. The straightforward program has been given in the introduction, we will concentrate on the calculation part, showing how to code our calculation to make it be efficient.

As explained in the introduction, the straightforward program is inefficient because of redundant repeated computation of *sum* and *length*. This inefficiency can be handled by tupling *steep*, *sum* and *length*, using the tupling transformation [9, 18]. The specific tupling transformation for *steep* is coded in the calculation part. We use matching to extract body structures from *steep*, *sum* and *length* respectively, and then glue them together inside code-generation brackets <>.

Expanding the relation between *steep* and *steepOpt*, our system will automatically give the following efficient program:

$$\begin{array}{lcl} \mathit{steep} & = & \lambda x.\,\mathbf{let}\ (st, s, l) = \mathit{steep}'\ x\ \mathbf{in}\ st \\ \mathit{steep}'\ [\,] & = & (\mathit{True}, 0, 0) \\ \mathit{steep}'\ (a : x) & = & \mathbf{let}\ (st, s, l) = \mathit{steep}'\ x \\ & & \mathbf{in} \\ & & \quad \mathbf{let} \\ & & \qquad x \oplus_1 (y, z) = (x > y) \wedge z \\ & & \qquad x \oplus_2 y = x + y \\ & & \qquad x \oplus_3 y = 1 + y \\ & & \qquad g\ s\ l = s/l \\ & & \quad \mathbf{in} \\ & & \qquad (a \oplus_1 (g\ s\ l, st), a \oplus_2 s, a \oplus_3 l) \end{array}$$

which is essentially the same as the efficient one given in the introduction. Note an alternative way to use **let** instead of explicit substitution when instantiating bound variables of $\oplus_i$'s and g in code generation.

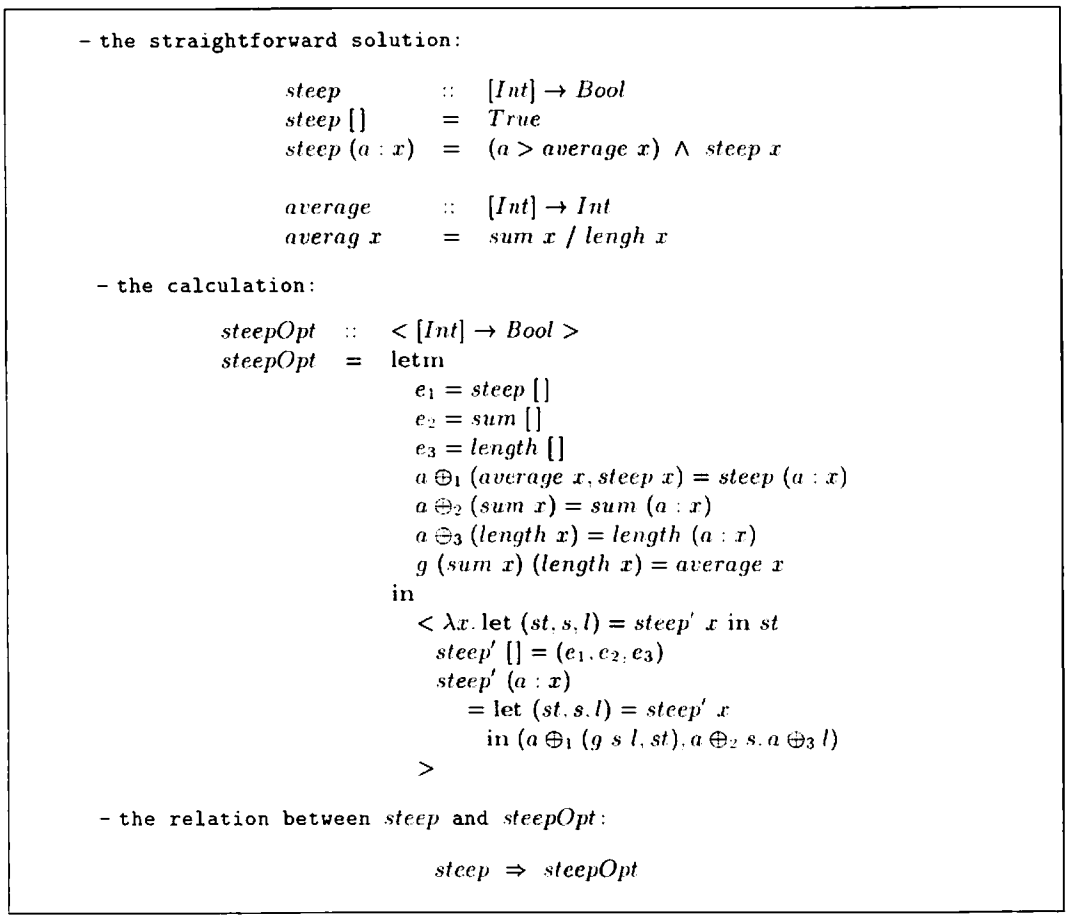


Figure 1. A calculation carrying program for the steep problem.

3 Formal Development

In this section, we give the formal definition of the core language in Figure 2 for writing calculation carrying programs, including its syntax, semantics and typing rules. Rather than inventing a completely new language, we tried our best to extend the existing functional languages as little as possible. The features of our language are two-fold.

- First, it is similar to existing functional languages like Haskell. The crucial difference is that patterns that are used in λ abstraction, let binding and **case** matching are extended to be higher order ones which may contain function variables. To make this more explicit, we introduce three new constructs, namely λ^m, **letm**, and **casem**.
- Second, it is a kind of meta language inspecting and generating codes, but with the requirement that *open* code containing free variables not be allowed to be generated. The advantage of doing so is that we can guarantee the type correctness of generated programs

3.1 Object Language: Coding Naive Solutions

The object language, as defined by the upper part in Figure 2, is nothing special but a subset of Haskell, a polymorphically typed pure functional language. We do not discuss their details. Rather we give more examples for the definitions of functions used so far or to be used later.

$$
\begin{array}{lcl}
sum & :: & [Int] \to Int \\
sum & = & \lambda x.\,\textbf{case}\ x\ \textbf{of} \\
 & & \quad [] \to 0 \\
 & & \quad (a : x) \to a + sum\ x \\
length & :: & [Int] \to Int \\
length & = & \lambda x.\,\textbf{case}\ x\ \textbf{of} \\
 & & \quad [] \to 0 \\
 & & \quad (a : x) \to 1 + length\ x
\end{array}
$$

Definition:

$$def \quad ::= \quad f = e \quad \text{function definition}$$

Expression:

e	::=	v	expression variable
	|	n	constant
	|	$\lambda v.\, e$	lambda abstraction
	|	$e_1\ e_2$	function application
	|	**case** e **of** $p_1 \rightarrow e_1; \ldots; p_n \rightarrow e_n$	case expression
	|	$< e >$	encode
	|	$e^{\$}$	decode
	|	em	expression with higher-order matching

Expression with Higher-order Matching:

em	::=	$\lambda^m < e_p > . e_b$	meta lambda
	|	**letm** $< e_p >=< e_b > \Leftarrow e_c$ **in** e	meta let
	|	**casem** $< e >$ **of** $< e_{p_1} > \rightarrow e_1; \ldots; < e_{p_n} > \rightarrow e_n$	meta case

Simple Pattern:

p	::=	v	pattern variable
	|	$C\ p_1\ \cdots\ p_n$	constructor pattern

Relation between Object and Meta Symbols:

$$bind \quad ::= \quad f \Rightarrow g \quad \text{binding between object and meta functions}$$

Figure 2. The core language

$$\begin{aligned} foldr &:: (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b \\ foldr &= \lambda(\oplus).\lambda e.\lambda x.\textbf{case } x \textbf{ of} \\ &\quad [\,] \rightarrow e \\ &\quad (a : x) \rightarrow a \oplus foldr\ (\oplus)\ e\ x \end{aligned}$$

For the sake of readability, we sometimes take liberty to use some familiar syntactic sugars like infix notations, pattern matching equations instead of case constructs, **let** binding instead of λ abstraction, and etc.

3.2 Meta Language: Coding Calculation

The meta language basically consists of two parts. One is for encoding and decoding expression, namely $< e >$ and $e^{\$}$. And the other is for inspecting expressions by higher order matching as used in λ^m, letm and casem. We explain each construct informally below, before giving the formal definition later.

- $< e >$

  It is used to build expression code (object expression). For instance, $< 2 + 3 >$ builds the code of $2 + 3$.

  Two remarks should be made. First, our system automatically flattens nested encodings. For example, $< 2 + < 3 + 4 >>$ is automatically transformed into $< 2 + (3 + 4) >$. This automatic decoding technique improves readability of programs; otherwise we need to explicitly add decoding $\_^{\$}$. Second, if e has some free variable like $< \lambda x.x + y >$, we require the free variables (y for this example) be bounded by an expression whose free variables again are all bounded. If y is bound by , for example, $y \mapsto 2+3$ then e is equivalent to the code with **let**: $<$ let $y = 2+3$ in $\lambda x.x + y >$.

- $e^{\$}$

  It is used for decoding, and is analogous to the *unquote* in Lisp. We have $< e >^{\$}$ evaluates to e.

- $\lambda^m e_p.\, e_b$

  It is used to define a meta function, which matches the input with e_p to bind the free variables in e_p, and then computes e_b. For example,

the fusion calculation rule in Section 2 can be specified by

$$fusion \;=\; \lambda^m < f \circ foldr\ (\oplus)\ e > .\ (\textbf{letm} \ldots \textbf{in} \ldots).$$

For readability, we sometimes write $f = \lambda^m e_p.\, e_b$ as $f\ e_p = e_b$.

Note that we require the pattern expression e_p (also that in the later letm and casem) be tight (i.e., has neither β nor η redex) to take advantage of the higher-order matching algorithm in [13].

- **letm** $e_p = e_b \Leftarrow e_c$ **in** e

  It is used to match expression object e_b with expression object e_p to bind free variables in e_p while satisfying the condition e_c, and then to compute e as its result. Note that e_c is useful to reduce the number of solutions[1] of bindings when matching e_b with e_p. With this construct, we can code a simple one-step hoisting transformation by

```
hoist1 :: <a> → <a>
hoist1 <e> =
  letm
    <c (let x = (let y = e0 in e1) in e2)>
      = <e> ⇐ not(y `isElem` fv e2)
  in
    <c (let y = e0 in (let x = e1 in e2))>
```

  If y is not free in e_2 under a context c, we can hoist y up. Here, $fv\ e_2$ computes all free variables in e_2.

  Since we know that e_b and e_p are expression objects, we may omit the brackets <> around them for readability.

```
hoist1 <e> =
  letm
    c (let x = (let y = e0 in e1) in e2) = e
      ⇐ not(y `isElem` fv e2)
  in
    <c (let y = e0 in (let x = e1 in e2))>
```

  In this paper, we often omit the brackets <> when this is clear from the context.

- **casem** e **of** $e_{p_1} \rightarrow e_1 ; \ldots ; e_{p_n} \rightarrow e_n$

  This construct provides a convenient way to specify manipulation of expression case by case. It matches expression object e with expression objects $e_{p_1}, \ldots, e_{p_n}$ one by one till matching succeeds, say at e_{p_i}, then it evaluate e_i. It should be noted that the result can be any expressions, not limiting to expression code. As an example, the following definition of *isContext*, determining whether an expression represents a context, gives a boolean as result.

```
isContext ::< a > → Bool
isContext = λ^m e. casem e of
    λx.e' → x `isElem` fv e'
    _ → False
```

  This reads that an expression object is said to be a context if it is a function with its bound variable appearing in the body as holes.

[1] Note that matching two expression terms may give many solutions, which is different from the pattern matching in functional languages [28].

3.3 Semantics

The semantics for the core language is quite similar to that of general functional languages except for the following two points. First, expressions evaluate to a *list* of values rather than a single one, because the core language allows higher order pattern matching which may compute *many* solutions. Second, expression codes (objects) are treated as first-class values, and can be manipulated in a similar way as other values like lists.

We shall take a close look at the higher-order matching we use, before giving the semantics of the language.

Higher-order Matching

Higher-order matching plays an important role in our meta language. Given two expression codes e_p (the pattern) and e_t (the term) in the object language, matching is to find all possible substitutions ϕ such that $\phi\ e_p = e_t$. Here equality is taken modulo renaming (α-conversion), elimination of redundant abstraction (η-conversion), and substitution of arguments for parameters (β-conversion). We call such ϕ a *match*, which is a map from variables to expression objects:

$$\phi :: \mathbf{Var} \rightarrow \mathbf{Exp}$$

where **Exp** denotes all expression.

Clearly it is undesirable that matching gives a potentially infinite set of matches in the context of automatic program transformation. In [22], Huet and Lang suggested restricting attention to matching of *second-order* terms, and gave a matching algorithm which is both *decidable* and *complete* to compute a finite number of incomparable matches. Recently, de Moor and Sittampalam [13] extended the second-order matching algorithm, and present a new one to suit transformation

of Haskell programs. The new algorithm can accept higher-order and polymorphically typed terms, sharing the property that it returns a well-defined, finite set of matches. We will not be involved in more detailed discussion on higher-order matching. Rather we use these results in this paper through the function *matching*:

$$matching :: \mathbf{Exp} \to \mathbf{Exp} \to [\mathbf{Var} \to \mathbf{Exp}]$$

which accepts two expressions e_p and e_t, and returns a list of matches. For instance,

$$matching\ (\lambda x.\, v\ x\ c)\ (\lambda x.\, c'\ x\ c)$$

gives two matches, namely

$$\{v \mapsto c'\} \quad \text{and} \quad \{v \mapsto \lambda x.\lambda y.\, c'\ x\ c\}.$$

Here c and c' represent some constants.

As it is required in [13] that the pattern e_p be free of η-redex, and that the term e_t be free of β and η redex, we use the function

$$reduce :: \mathbf{Exp} \to \mathbf{Exp}$$

to reduce β and η redex in an expression.

Interpretation

To interpret both object and meta expressions, we define $\mathbf{Val}^+$ by extending the ordinary value domain **Val** with expression codes **Exp** such that expressions can be manipulated in a similar way as other values like integers. We use **Env** for the environment, and **Val** for the values. The environment maps a variable to a value:

$$\mathbf{Env} \to \mathbf{Val}^+.$$

Figure 3 gives a formal definition of the semantics of the core language. For simplicity, we have assumed that each bound variable has been renamed with a unique name, and hence we do not consider name conflicts in definition of semantics.

$\mathcal{E}$ evaluates a meta expression, under an environment, to a *list* of values instead of a single one. This is due to many possible matches for a single matching. Using list here resembles the technique used to deal with nondeterminism in construction of the monadic parser [24]. $\sqcup$ flattens a list of lists, *unnest* is to decode nested encodings, and *introLet* $\rho\ e$ is to construct code by expanding environment using *let* as seen the discussion of encoding in Section 3.2. $\rho_1 \oplus \rho_2$, for extending environment ρ_1 with ρ_2, is defined by

$$\begin{aligned}(\rho_1 \oplus \rho_2)\ x &= \rho_2\ x, \quad x \text{ is defined in } \rho_2 \\ &= \rho_1\ x, \quad \text{otherwise.}\end{aligned}$$

The main characteristics of $\mathcal{E}$ is that it binds local variables using matching. With this in mind, it should not be difficult to understand its definition. Notice the intensive use of list comprehension for defining $\mathcal{E}$.

Now, the meaning of $f \Rightarrow f'$ is simple, just to associate f with one of the codes from computation of $\mathcal{E}[\![f']\!]\rho$.

3.4 Type System

In typing, there is no big difference between the higher-order matching in pure language and the ordinary matching in traditional functional languages. To handle meta codes, we extend the traditional type system to include the type to represent codes. For more details, please see [19]. Since we do not allow to generate open code, we can thus treat variables in meta codes in the same way as in other places, and easily guarantee the correctness of generated codes.

4 Applications

In Section 2, we have illustrated the idea of calculation carrying programs by two simple examples. We will give more practical application examples in this section. Specifically, we will demonstrate in general how to declare calculational rules , calculational strategies, as well as program development process.

4.1 Coding Calculation Rules

Rules (laws) are most fundamental to transform expressions. In our language they are naturally described by using higher-order matching. Consider to define the rule transforming $0 + x$ to x. We can simply code it as

$$\begin{aligned}removeZero &:: < Int > \to < Int > \\ removeZero &= \lambda^m\ < 0 + x > .\ < x > .\end{aligned}$$

For readability, we often write the above as:

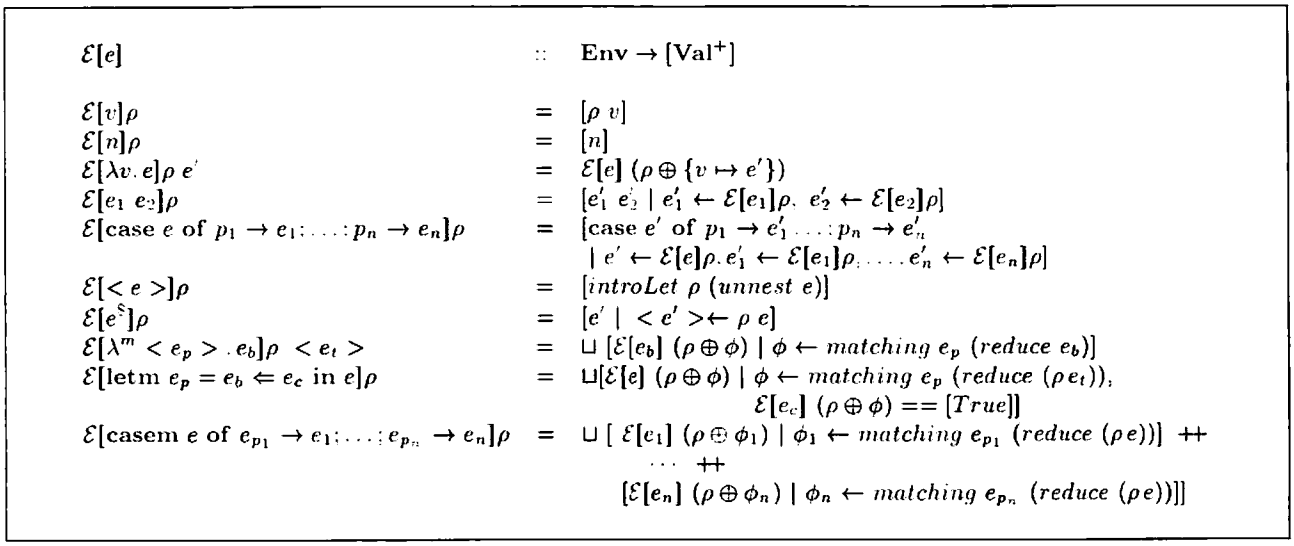
$$removeZero\ < 0 + x > \ = \ < x > .$$

Two remarks are worth making. First, the expressions to be matched can have multiple occurrences of the same variable [22], i.e., the rules are not necessary to be left linear. So,

$$\begin{aligned}sum2Double &:: Num\ a \Rightarrow < a > \to < a > \\ sum2Double &= \lambda^m < x + x > .\ < 2 * x >\end{aligned}$$

is a valid rule.

Second, like programming in ordinary pattern matching, we can define calculation in a case-by-case



$$
\begin{array}{lcl}
\mathcal{E}[e] & :: & \mathbf{Env} \rightarrow [\mathbf{Val}^+] \\
\\
\mathcal{E}[v]\rho & = & [\rho\ v] \\
\mathcal{E}[n]\rho & = & [n] \\
\mathcal{E}[\lambda v.e]\rho\ e' & = & \mathcal{E}[e]\ (\rho \oplus \{v \mapsto e'\}) \\
\mathcal{E}[e_1\ e_2]\rho & = & [e_1'\ e_2' \mid e_1' \leftarrow \mathcal{E}[e_1]\rho,\ e_2' \leftarrow \mathcal{E}[e_2]\rho] \\
\mathcal{E}[\mathbf{case}\ e\ \mathbf{of}\ p_1 \rightarrow e_1; \ldots; p_n \rightarrow e_n]\rho & = & [\mathbf{case}\ e'\ \mathbf{of}\ p_1 \rightarrow e_1' \ldots; p_n \rightarrow e_n' \\
 & & \quad \mid e' \leftarrow \mathcal{E}[e]\rho, e_1' \leftarrow \mathcal{E}[e_1]\rho, \ldots, e_n' \leftarrow \mathcal{E}[e_n]\rho] \\
\mathcal{E}[< e >]\rho & = & [introLet\ \rho\ (unnest\ e)] \\
\mathcal{E}[e^{\$}]\rho & = & [e' \mid < e' > \leftarrow \rho\ e] \\
\mathcal{E}[\lambda^m < e_p > .e_b]\rho\ < e_t > & = & \sqcup\ [\mathcal{E}[e_b]\ (\rho \oplus \phi) \mid \phi \leftarrow matching\ e_p\ (reduce\ e_b)] \\
\mathcal{E}[\mathbf{letm}\ e_p = e_b \Leftarrow e_c\ \mathbf{in}\ e]\rho & = & \sqcup[\mathcal{E}[e]\ (\rho \oplus \phi) \mid \phi \leftarrow matching\ e_p\ (reduce\ (\rho\, e_t)), \\
 & & \qquad \mathcal{E}[e_c]\ (\rho \oplus \phi) == [True]] \\
\mathcal{E}[\mathbf{casem}\ e\ \mathbf{of}\ e_{p_1} \rightarrow e_1; \ldots; e_{p_n} \rightarrow e_n]\rho & = & \sqcup\ [\ \mathcal{E}[e_1]\ (\rho \oplus \phi_1) \mid \phi_1 \leftarrow matching\ e_{p_1}\ (reduce\ (\rho e))]\ +\!\!+ \\
 & & \quad \cdots\ +\!\!+ \\
 & & \quad [\mathcal{E}[e_n]\ (\rho \oplus \phi_n) \mid \phi_n \leftarrow matching\ e_{p_n}\ (reduce\ (\rho e))]]
\end{array}
$$

Figure 3. The semantics of the core expressions

way. For example, the following *concatRm* are used to remove the concatenation operator "++" by matching for two cases:

$$
\begin{array}{l}
concatRm\ ::\ < [a] > \rightarrow < [a] > \\
concatRm\ < [\,] +\!\!+ x > \\
\quad = \ < x > \\
concatRm\ < (a : x) +\!\!+ y > \\
\quad = \ < a : (concatRm\ < x +\!\!+ y >)^{\$} >
\end{array}
$$

This is equivalent to

$$
\begin{array}{l}
concatRm = \\
\quad \lambda^m < e > .\,\mathbf{casem}\ e\ \mathbf{of} \\
\quad [\,] +\!\!+ x \quad \rightarrow \ < x > \\
\quad (a : x) +\!\!+ y \ \rightarrow \ < a : (concatRm\ < x +\!\!+ y >)^{\$} > .
\end{array}
$$

Given an expression, we will match it in the order of the cases given. Another small example is

$$
\begin{array}{l}
simpleIf\ ::\ < a > \rightarrow < a > \\
simpleIf\ < \mathbf{if}\ True\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2 > = e_1 \\
simpleIf\ < \mathbf{if}\ Flase\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2 > = e_2
\end{array}
$$

which performs static evaluation of **if** expression.

Following this way, it should not be difficult to code other interesting calculation laws such as the tupling law [18], parallelizing theorem [20], the accumulation calculation theorem [17], and the diffusion calculation law [21].

4.2 Coding Calculation Strategies

Valid transformations on program can be described by a set of calculation rules; while calculation strategies are applied to obtain the desired optimization effects [33]. In this section, we would like to demonstrate that those important strategies in [33] can be programmed here in a more concise and direct way.

Basically, the sequential application of two rules r_1 and r_2 to a term $< t >$ can be coded by $r_2\ (r_1\ < t >)$, and the choice application of rules $r_1, \ldots, r_n$ to term $< t >$ can be coded something like **casem** t **of** $case_1 \rightarrow r_1\ t; \cdots; case_n \rightarrow r_n\ t$, as we have shown in the definition of *concatRm*.

To appreciate its practical use, suppose we want to apply the fusion calculation *fusion* to an expression to remove as many function compositions as possible. To this end, we repeatedly select the fusible subexpressions and apply the fusion calculation to them, until we have no fusible subexpressions any more.

$$
\begin{array}{l}
applyFusion\ ::\ < a > \rightarrow < a > \\
applyFusion\ p@ < f \circ g > = \\
\quad \mathbf{casem}\ (applyFusion\ g)\ (a : x) > \ \mathbf{of} \\
\quad\quad a \oplus\ (g\ x) \rightarrow fusion\ < (applyFusion\ f)^{\$} \circ \\
\quad\quad\quad\quad\quad foldr\ (\oplus)\ (g\ [\,]) > \\
\quad\quad \_ \rightarrow\ p \\
applyFusion\ < c\ (\lambda x.(f\ (g\ x))) > = \\
\quad \mathbf{let} \\
\quad\quad c' = applyFusion\ c \\
\quad\quad fg = applyFusion\ < f \circ g > \\
\quad \mathbf{in}\ < c'\ fg > \\
applyFusion\ e = e
\end{array}
$$

4.3 Coding Programming Development

The creative steps in transformational programming are very difficult to be implemented in a fully automatic

way. The calculation carrying code provides us with a flexible means to code these creative steps.

Consider we want to develop an efficient program to reverse a list. A naive definition is

$$\begin{array}{lcl} rev & :: & [a] \to [a] \\ rev\ [\,] & = & [\,] \\ rev\ (a : x) & = & rev\ x +\!\!+ [a] \end{array}$$

which is a quadratic program. It has been shown in [23] that we need two creative steps for this improvement. First, we need to introduce an accumulation parameter starting from [], as define by

$$\begin{array}{lcl} rev\ x & = & fastrev\ x\ [\,] \\ fastrev\ x\ y & = & rev\ x +\!\!+ y. \end{array}$$

Second, to derive efficient definition for *fastrev*, we need to apply the fusion calculation to $(+\!\!+\ y) \circ reverse$, where we are required to use the associative property of $+\!\!+$:

$$(x +\!\!+ y) +\!\!+ z\ =\ x +\!\!+ (y +\!\!+ z).$$

Although these two steps are difficult to be made automatic in general, we are able to code them as

$$\begin{array}{lcl} revOpt & :: & < [a] \to [a] > \to < [a] \to [a] > \\ revOpt & = & \mathbf{letm}\ e = [\,] +\!\!+ y \\ & & \qquad a \oplus (x +\!\!+ y) = \\ & & \qquad\quad assoc\ < rev\ (a : x) +\!\!+ y > \\ & & \mathbf{in}\ < (\lambda y.\ foldr\ (\oplus)\ e)\ [\,] > \end{array}$$

where *assoc* specifies the association rule,as defined by:

$$\begin{array}{l} assoc :: < a > \ \to \ < a > \\ assoc\ < (x +\!\!+ y) +\!\!+ z > = < x +\!\!+ (y +\!\!+ z) > . \end{array}$$

5 Related Work

Incorporating the use of higher order matching (or unification) to concisely express program transformations have been seen in many systems, such as Ergo [29], KORSO [6] and MAG [12]. Our work has received much influence from the MAG system which is designed to support generic transformational programming. However, MAG uses higher order matching only *implicit* by the system. In contrast, we design a language with general matching mechanism so that programmers can express explicitly what exactly they want to match.

There are a number of specialized *pattern languages* for the purpose of program inspection and transformation [16, 1, 26, 10, 3]. Often, these do not include higher order patterns. Instead, they provide a set of primitives for matching and building subtrees, but for the most part require that tree traversal be programmed explicitly in *imperative* style, node by node.

Our work is also related to the work on meta programming. MetaML [31] is a statically-typed multi-stage programming language, allowing the programmer to construct, combine, and execute code fragments. But using MetaML to code calculation would be difficult and complicated, because it does not have powerful matching mechanism.

The system RML optimizer [33] is close in spirit to our system. It studies how to transform the source code of a program into another one in the compiler for optimization, based on ideas from term rewriting. It argues that a good way is to use explicit specification of rewriting strategies, and it shows that it is possible to break down rewrite rules into two primitives: *matching* against term patterns and *building* terms. Unlike our system, all these are done inside compiler rather than being open to programmers.

There are a lot of work on developing fast implementation of matching algorithm [8, 11, 13]. Particularly the Oxford Programming Tools Group are now actively researching on this topic.

6 Concluding Remarks

We have proposed a new framework for writing calculation carrying programs, for the purpose of relaxing the tension between clarity and efficiency in programming. As demonstrated by several convincing examples, this framework has shown its significance and promise both in support of transformational/calculational programming and in compiler construction based on program transformation.

It is worth noting again that we do not need folding in our transformation at all. This is in sharp contrast to many existing transformation systems that are based on fold/unfold transformation technique. We believe that the partial correctness problem and high implementation cost of fold / unfold transformation prevent it from transforming large scale programs.

We are investigating more on higher-order meta functions and modularity of calculations, both of which have not been fully addressed in this paper. It might be interesting to port it to Gofer to make it be really Good for expressing reasoning.

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