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# Deriving Structural Hylomorphisms From Recursive Definitions

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## Summary.

In functional programming, small programs are often “glued” together to construct a complex program. Program fusion is an optimizing process whereby these small programs are fused into a single one and intermediate data structures are removed. Recent work has made it clear that the process is especially successful if the recursive definitions are expressed in terms of hylomorphisms. In this paper, we propose an algorithm which can automatically turn all practical recursive definitions into structural hylomorphisms making program fusion be easily applied.

## 1 Introduction

The compositional style of functional programming has many advantages of clarity and higher level of modularity. It constructs a complex program by gluing components which are relatively simple, easier to write, and potentially more reusable. However, some data structures, which are constructed in one component and consumed in another but never appear in the result of the whole program, give rise to the problem of efficiency.

Consider a toy example of function *all* which tests whether all the elements of a list satisfy the given predicate *p*. It may be defined as follows.

$$\begin{aligned} \text{all } p &= \text{and} \circ \text{map } p \\ \text{where } \text{and} &= \lambda xs. \text{case } xs \text{ of } Nil \rightarrow True; Cons(a, as) \rightarrow a \wedge (\text{and } as) \end{aligned}$$

Here *p* is applied to all the elements of the list producing an intermediate list of Booleans which are then “anded” together by the function *and* producing a single Boolean result. To make the function *all* be computed efficiently, it is expected to fuse *and* and *map p* together to have the following new definition where the intermediate list of Booleans is not produced.

$$\text{all } p = \lambda xs. \text{case } xs \text{ of } Nil \rightarrow True; Cons(a, as) \rightarrow pa \wedge (\text{all } as)$$

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There are two kinds of approaches dealing with such fusion. One, first proposed by Wadler as called *deforestation*, aims to fuse *arbitrary* functions by *fold-unfold* transformations, keeping track of function calls and using clever control to avoid infinite unfolding[Chi92, Wad88]. The other, quite differently, makes use of some *specific* forms such as *catamorphisms* (or called *folds*), *anamorphisms* (or called *unfolds*) and *hylomorphisms* and finds how they interact[GLJ93, SF93, TM95].

The second approach has been proved to be more practical in a real implementation in compilers, although at first sight it seems less general than the former. Its theoretical basis can be found in the study of *Constructive Algorithmics*[Fok92, Mal90, MFP91] which will be outlined in Section 2. In constructive algorithmics, data types are categorically defined as initial fixed points of functors, and functions from one data type to another are represented as structure-preserving maps between algebras. By doing so, an orderly structure can be imposed on the program and such structure can be exploited to facilitate program fusion.

However, this approach imposes recursive structures of specific forms on programs, which is unrealistic in practical functional programming. One attempt has been made by Launchbury and Sheard[LS95]. They gave an algorithm to turn recursive definitions into so-called *build-cata* forms (i.e. catamorphisms with constructors being parameterized) so that the *shortcut deforestation* technique becomes applicable. One major problem still left is that the build-cata form is too restrictive to describe some kinds of practical recursive definitions and therefore many intermediate data structures cannot be removed.

The purpose of this paper is to demonstrate how practical recursive definitions can be automatically turned into hylomorphism, general forms covering all those in [MFP91, SF93, TM95], which makes program fusion transformation be applied better. The main contribution of this work is as follows.

- We propose an algorithm that can automatically turn almost *all* recursive definitions to structural hylomorphisms. With the use of fusion systems[SF93, KL94, TM95] successfully developed for hylomorphisms, we can improve a larger class of programs freely defined by programmers.
- Our algorithm is guaranteed to be correct, and to terminate with a successful hylomorphism. To the contrary, Launchbury and Sheard's derivation algorithm of build-catas from recursive definitions may fail to give build-catas[LS95]. Their algorithm has to be on the alert against failure of the "two-stage fusion" and gives up the derivation in case failure occurs.
- Our algorithm structures hylomorphisms so that the Hylo Fusion and Acid Rain Theorems (Section 2) can be effectively applied. Particularly, we propose a new theorem for deriving polymorphic functions (Section 5.2), namely  $\tau$  and  $\sigma$ , for the use of Acid Rain Theorem.
- Our algorithm does not limit its use to the fusion of recursions inducting over a single data structure. It is helpful for the fusion of recursions over multiple data structures without introducing new fusion theorems as in [FSZ94] (Section 4.4). Moreover, it is also useful for other program optimizations

such as removal of multiple traversal over the same data structures [Tak87]. This is because general optimization rules are much easier to be defined over hylomorphisms rather than over the recursions.

This paper is organized as follows. In Section 2 we review the previous work in Constructive Algorithmics, the theoretical basis of hylomorphisms. Section 3 defines a simple language for the description of recursive definitions. We discuss our algorithm in Section 4 and 5. In Section 4 we define our algorithm for deriving hylomorphisms from recursive definitions, and in Section 5 we give the algorithm for structuring hylomorphisms so that the two fusion theorems can be effectively applied. Section 6 discusses the related work and Section 7 gives the conclusion.

## 2 Background

In this section, we review previous work on constructive algorithmics and explain some basic facts which provide theoretical basis of our method. Throughout this paper, our default category  $C$  is a  $\mathcal{CPO}$ , the category of complete partial orders with continuous functions.

### 2.1 Functors

Endofunctors on category  $C$  (functors from  $C$  to  $C$ ) are used to capture both data structure and control structure in a type definition. In this paper, we assume that all the data types are defined by endofunctors which are only built up by the following four basic functors. Such endofunctors are known as *polynomial functors*.

**Definition 1 (Identity)** The identity functor  $I$  on type  $X$  and its operation on functions are defined as follows.

$$I X = X, I f = f \quad \square$$

**Definition 2 (Constant)** The constant functor  $!A$  on type  $X$  and its operation on functions are defined as follows.

$$!A X = A, !A f = id$$

where  $id$  stands for the identity function.  $\square$

**Definition 3 (Product)** The product  $X \times Y$  of two types  $X$  and  $Y$  and its operation to functions are defined as follows.

$$\begin{aligned} X \times Y &= \{(x, y) \mid x \in X, y \in Y\} \\ (f \times g) (x, y) &= (f x, g y) \end{aligned}$$

Some related operators are:

$$\begin{aligned} \pi_1 (a, b) &= a, \quad \pi_2 (a, b) = b \\ (f \triangle g) a &= (f a, g a). \end{aligned} \quad \square$$

**Definition 4 (Separated Sum)** The separated sum  $X + Y$  of two types  $X$  and  $Y$  and its operation to functions are defined as follows.

$$\begin{aligned} X + Y &= \{1\} \times X \cup \{2\} \times Y \\ (f + g) (1, x) &= (1, f x) \\ (f + g) (2, y) &= (2, g y) \end{aligned}$$

Some related operators are:

$$\begin{aligned} \iota_1 a &= (1, a), \quad \iota_2 b = (2, b) \\ (f \nabla g) (1, x) &= f x \\ (f \nabla g) (2, y) &= g y. \end{aligned} \quad \square$$

Although the product and the separated sum are defined over 2 parameters, they can be naturally extended for  $n$  parameters. For example, the separated sum over  $n$  parameters can be defined by  $+_{i=1}^n X_i = \cup_{i=1}^n \{i\} \times X_i$  and  $(+_{i=1}^n f_i) (j, x) = (j, f_j x)$ .

## 2.2 Data Types as Initial Fixed Points of Functors

A data type is a collection of operations (data constructors) denoting how each element of the data type can be constructed in a finite way, and via these data constructors functions on the type may be defined. So a data type is a particular algebra, one distinguished property is categorically known as initiality of the algebra. Let  $C$  be a category and  $F$  be an endofunctor from  $C$  to  $C$ .

**Definition 5 ( $F$ -algebra)** An  $F$ -algebra is a pair  $(X, \phi)$ , where  $X$  is an object in  $C$ , called the carrier of the algebra, and  $\phi$  is a morphism from object  $F X$  to object  $X$  denoted by  $\phi :: F X \rightarrow X$ , called the operation of the algebra.  $\square$

**Definition 6 ( $F$ -homomorphism)** Given  $F$ -algebras  $(X, \phi)$  and  $(Y, \psi)$ , the  $F$  homomorphism from  $(X, \phi)$  to  $(Y, \psi)$  is a morphism  $h$  from object  $X$  to object  $Y$  in category  $C$  satisfying  $h \circ \phi = \psi \circ F h$ .  $\square$

**Definition 7 (Category of  $F$ -algebras)** The category of  $F$ -algebras has as its objects the  $F$ -algebras and has as its morphisms all  $F$ -homomorphisms between  $F$ -algebras. Composition in the category of  $F$ -algebra is taken from  $C$ , and so are the identities.  $\square$

It is known that an initial object in the category of  $F$ -algebras exists when  $F$  is a polynomial functors [Mal90]. The representative for the initial algebra is denoted by  $\mu F$ . Let  $(T, in_F) = \mu F$ ,  $\mu F$  defines a data type  $T$  with the data constructor  $in_F : F T \rightarrow T$ . Function  $out_F : T \rightarrow F T$  is the inverse of  $in_F$  and it destructs its argument and is therefore called data destructor. To be concrete, consider the data type of cons lists given by the following equation with elements of type  $A$ :

$$List A = Nil \mid Cons (A, List A).$$

It is categorically defined as the initial object of  $(List a, Nil \nabla Cons)^1$  in the category of  $L_A$ -algebras, where  $L_A$  is the endofunctor defined by  $L_A = !1 + !A \times I$  ( $1$  is

<sup>1</sup> Strictly speaking, the  $Nil$  should be written as  $\lambda().Nil$ . In this paper, the function with the form of  $\lambda().t$  will be simply denoted as  $t$ .

the terminal object in  $C$ ). Here, the data constructor and the data destructor are as follows.

$$\begin{aligned} in_{LA} &= Nil \nabla Cons \\ out_{LA} &= \lambda xs. \text{ case } xs \text{ of } Nil \rightarrow (1, ()); Cons (a, as) \rightarrow (2, (a, as)) \end{aligned}$$

### 2.3 Hylomorphisms over data types

Hylomorphisms in triplet form[TM95] are defined as follows.

**Definition 8 (Hylomorphism in triplet form)** Given morphisms  $\phi : GA \rightarrow A$ ,  $\psi : B \rightarrow FB$  and natural transformation  $\eta : F \rightarrow G$ , the hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G,F}$  is defined as the the least morphism  $f : B \rightarrow A$  satisfying the following equation.

$$f = \phi \circ (\eta \circ F f) \circ \psi \quad \square$$

Hylomorphisms are powerful in description in that practically every recursion of interest can be specified by hylomorphisms[BdM94]. Hylomorphisms are quite general in that many useful forms are their special cases as defined below. Note that we sometimes omit the subscripts  $G$  and  $F$  when it is clear from the context.

#### Definition 9 (Catamorphism, Anamorphism, Map)

Let  $(T_F, in_F) = \mu F$ ,  $(T_G, in_G) = \mu G$ .

$$\begin{aligned} \llbracket \_ \rrbracket_F &: \forall A. (F A \rightarrow A) \rightarrow T_F \rightarrow A \\ \llbracket \phi \rrbracket_F &= \llbracket \phi, id, out_F \rrbracket_{F,F} \end{aligned}$$

$$\begin{aligned} \llbracket \_ \rrbracket_{F,F} &: \forall A. (A \rightarrow F A) \rightarrow A \rightarrow T_F \\ \llbracket \psi \rrbracket_{F,F} &= \llbracket in_F, id, \psi \rrbracket_{F,F} \end{aligned}$$

$$\begin{aligned} \llbracket \_ \rrbracket_{G,F} &: (F \rightarrow G) \rightarrow T_F \rightarrow T_G \\ \llbracket \eta \rrbracket_{G,F} &= \llbracket in_G, \eta, out_F \rrbracket_{G,F} \end{aligned} \quad \square$$

Catamorphisms  $\llbracket \_ \rrbracket$  are generalized foldr (or reduces) operators that substitute the constructor of a data type with other operation of the same signature. Dually, anamorphisms  $\llbracket \_ \rrbracket$  are generalized unfold (or generations) operators. Maps  $\llbracket \_ \rrbracket$  apply a natural transformation on the data structure. Maps can be represented as both catamorphisms and anamorphisms based on the following Hylo Shift Theorem.

#### Theorem 1 (Hylo Shift)

$$\llbracket \phi, \eta, \psi \rrbracket_{G,F} = \llbracket \phi \circ \eta, id, \psi \rrbracket_{F,F} = \llbracket \phi, id, \eta \circ \psi \rrbracket_{G,G}. \quad \square$$

The Hylo Shift Theorem shows that some computations can be shifted within a hylomorphism. For program fusion, hylomorphisms possess the general laws called the *Hylo Fusion Theorem*.

#### Theorem 2 (Hylo Fusion)

$$\begin{aligned} \text{Left Fusion Law: } f \circ \phi &= \phi' \circ G f \implies f \circ \llbracket \phi, \eta, \psi \rrbracket_{G,F} = \llbracket \phi', \eta, \psi \rrbracket_{G,F} \\ \text{Right Fusion Law: } \psi \circ g &= F g \circ \psi' \implies \llbracket \phi, \eta, \psi \rrbracket_{G,F} \circ g = \llbracket \phi, \eta, \psi' \rrbracket_{G,F} \end{aligned} \quad \square$$

$Decl$	$::= v = b$	(recursive) function definition
$b$	$::= \lambda v_s. \text{ case } t_0 \text{ of } r$	definition body
$v_s$	$::= v \mid (v_1, \dots, v_n)$	argument
$r$	$::= p_1 \rightarrow t_1; \dots; p_n \rightarrow t_n$	alternatives
$t$	$::= v$	variable
	$\mid (t_1, \dots, t_n)$	term tuple
	$\mid v t$	function application
	$\mid C t$	constructor application
$p$	$::= C p$	pattern
	$\mid (p_1, \dots, p_n)$	pattern tuple
	$\mid v$	variable

**Fig. 1** The language for Recursive Definitions

These laws are quite general in the sense the functions to be fused with, e.g.,  $f$  and  $g$  in Theorem 2, can be any functions. If  $f$  and  $g$  are restricted to specific hylomorphisms, we could have the following simple but practical *Acid Rain Theorem*[TM95].

**Theorem 3 (Acid Rain)**

$$\text{Cata-Hylo Fusion Law: } \frac{\tau : \forall A. (F A \rightarrow A) \rightarrow F' A \rightarrow A}{\llbracket \phi, \eta_1, \text{out}_F \rrbracket_{G,F} \circ \llbracket \tau \text{in}_F, \eta_2, \psi \rrbracket_{F',L} = \llbracket \tau(\phi \circ \eta_1), \eta_2, \psi \rrbracket_{F',L}}$$

$$\text{Hylo-Ana Fusion Law: } \frac{\sigma : \forall A. (A \rightarrow F A) \rightarrow A \rightarrow F' A}{\llbracket \phi, \eta_1, \sigma \text{out}_F \rrbracket_{G,F'} \circ \llbracket \text{in}_F, \eta_2, \psi \rrbracket_{F,L} = \llbracket \phi, \eta_1, \sigma(\eta_2 \circ \psi) \rrbracket_{G,F'}}$$

□

### 3 Language

To demonstrate our techniques, we use the language given in Fig. 1 for the description of *recursive* definitions. It is nothing special except that functions are defined in an un-curried way. In order to simplify our presentation, we restrict ourselves to single-recursive data types and functions without mutual recursions, since the standard tupling technique can transform mutual recursive definitions to non-mutual ones. We also assume that recursive function calls are not nested and only occur in the terms of the alternatives in the definition body.

Several simple examples of recursive definitions are given below. The function *sum* sums up all the elements in a list, the function *upto* generates a list of natural numbers between two given numbers, and the function *zip* turns a pair of lists into a list of pairs.

$$\begin{aligned} \text{sum} &: List\ a \rightarrow Int \\ \text{sum} &= \lambda xs. \text{ case } xs \text{ of} \\ &\quad Nil \rightarrow 0; \\ &\quad Cons(a, as) \rightarrow plus(a, \text{sum } as) \end{aligned}$$

$$\begin{aligned}
upto &: Int \times Int \rightarrow List Int \\
upto &= \lambda(b, e). \text{ case } b < e \text{ of} \\
&\quad True \rightarrow Nil; \\
&\quad False \rightarrow Cons(b, upto(plus(b, 1), e)) \\
zip &: List A \times List B \rightarrow List (A, B) \\
zip &= \lambda(xs, ys). \text{ case } (xs, ys) \text{ of} \\
&\quad (Nil, _) \rightarrow Nil; \\
&\quad (Cons(a, as), Nil) \rightarrow Nil; \\
&\quad (Cons(a, as), Cons(b, bs)) \rightarrow Cons((a, b), zip(as, bs))
\end{aligned}$$

Here *plus* is the function adding two integers. Next, let's consider definitions of higher order functions, such as

$$\begin{aligned}
map &: (A \rightarrow B) \rightarrow List A \rightarrow List B \\
map\ g &= \lambda xs. \text{ case } xs \text{ of} \\
&\quad Nil \rightarrow Nil; \\
&\quad Cons(a, as) \rightarrow Cons(g\ a, map\ g\ as)
\end{aligned}$$

which is not a valid definition in our language since the defined function *map g* is not a variable. The simplest way to solve this problem is to consider *map g* as a “packed” variable and to consider *g* as a global function. That is, the above definition is considered something like

$$\begin{aligned}
map\_g &= \lambda xs. \text{ case } xs \text{ of} \\
&\quad Nil \rightarrow Nil; \\
&\quad Cons(a, as) \rightarrow Cons(g\ a, map\_g\ as).
\end{aligned}$$

Viewed in this way, *map g* can be regarded as a valid definition. Below is another similar example.

$$\begin{aligned}
foldr1 &: (B \times C \rightarrow C) \rightarrow List B \rightarrow C \\
foldr1 \oplus &= \lambda xs. \text{ case } xs \text{ of } Nil \rightarrow error; \\
&\quad Cons(a, Nil) \rightarrow a; \\
&\quad Cons(a, as) \rightarrow a \oplus (foldr1 \oplus as)
\end{aligned}$$

## 4 Expressing Recursions as Hylomorphisms

In this section, we propose an algorithm to turn recursive definitions into hylomorphisms. Consider the following typical recursive definition of function *f*.

$$f = \lambda v_s. \text{ case } t_0 \text{ of } p_1 \rightarrow t_1; \dots; p_n \rightarrow t_n$$

If we can transform the right hand side to  $\phi \circ F f \circ \psi$ , it soon follows that  $f = \llbracket \phi, id, \psi \rrbracket_{F, F}$  from the definition of hylomorphisms.

### 4.1 Main idea

The trick to do so is to turn each term  $t_i$  ( $i = 1, \dots, n$ ) into  $g_i t'_i$ , a suitable function being applied to a new term. Put it in more detail, suppose that we have got that  $t_i = g_i t'_i$ , the original definition becomes:

$$f = \lambda v_s. \text{ case } t_0 \text{ of } p_1 \rightarrow g_1 t'_1; \dots; p_n \rightarrow g_n t'_n.$$

Extracting all  $g_i$ 's out and adding tag  $i$  to  $t'_i$  can give a compositional description of  $f$ :

$$f = (g_1 \nabla \cdots \nabla g_n) \circ (\lambda v_s. \text{case } t_0 \text{ of } p_1 \rightarrow (1, t'_1); \cdots; p_n \rightarrow (n, t'_n)).$$

If  $g_i$  can be expressed as  $\phi_i \circ F_i f$  where  $F_i$  is some functor, it soon follows that

$$g_1 \nabla \cdots \nabla g_n = (\phi_1 \nabla \cdots \nabla \phi_n) \circ (F_1 + \cdots + F_n) f.$$

Now replacing it in the above compositional description of  $f$  gives:

$$f = (\phi_1 \nabla \cdots \nabla \phi_n) \circ (F_1 + \cdots + F_n) f \circ (\lambda v_s. \text{case } t_0 \text{ of } p_1 \rightarrow (1, t'_1); \cdots; p_n \rightarrow (n, t'_n)).$$

According to the definition of hylomorphisms, we have  $f = \llbracket \phi, id, \psi \rrbracket_{F,F}$ , where

$$\begin{aligned} F &= F_1 + \cdots + F_n \\ \phi &= \phi_1 \nabla \cdots \nabla \phi_n \\ \psi &= \lambda v_s. \text{case } t_0 \text{ of } p_1 \rightarrow (1, t'_1); \cdots; p_n \rightarrow (n, t'_n). \end{aligned}$$

The essential point of our algorithm is, therefore, to derive a function  $\phi_i$ , a functor  $F_i$  and a new term  $t'_i$  from each term  $t_i$  satisfying  $t_i = (\phi_i \circ F_i f) t'_i$ .

#### 4.2 Deriving $\phi_i$ , $F_i$ and $t'_i$ from $t_i$

Our algorithm to derive  $\phi_i$ ,  $F_i$  and  $t'_i$  from  $t_i$  is informally as follows.

1. Identify all the occurrences of recursive calls in  $t_i$ , say they are  $f t_{i_1}, \cdots, f t_{i_{k_i}}$ ;
2. Find all the free variables in  $t_i$  bound in the definition except in  $t_{i_1}, \cdots, t_{i_{k_i}}$ , say they are  $v_{i_1}, \cdots, v_{i_{k_i}}$ ;
3. Define  $t'_i$  by tupling all the arguments of the recursive calls obtained in Step 1 and the free variables obtained in step 2, i.e.  $t'_i = (v_{i_1}, \cdots, v_{i_{k_i}}, t_{i_1}, \cdots, t_{i_{k_i}})$ .
4. Define  $F_i$  according to the construction of  $t'_i$  by  $F_i = !\Gamma(v_{i_1}) \times \cdots \times !\Gamma(v_{i_{k_i}}) \times I_1 \times \cdots \times I_{k_i}$ , where  $I_1 = \cdots = I_{k_i} = I$  and  $\Gamma$  returns the type of the given variable.
5. Define  $\phi_i$  by abstracting all recursive function calls in  $t_i$  by

$$\phi_i = \lambda(v_{i_1}, \cdots, v_{i_{k_i}}, v'_{i_1}, \cdots, v'_{i_{k_i}}). t_i[f t_{i_1} \mapsto v'_{i_1}, \cdots, f t_{i_{k_i}} \mapsto v'_{i_{k_i}}]$$

where  $v'_{i_1}, \cdots, v'_{i_{k_i}}$  are new variables used for replacing those occurrences of recursive calls  $f t_{i_1}, \cdots, f t_{i_{k_i}}$ .

#### 4.3 Algorithm for Deriving Hylomorphisms

The derivation algorithm described above is summarized in Fig. 2. The main algorithm is  $\mathcal{A}$  which turns a recursive definition into a hylomorphism. The algorithm  $\mathcal{A}$  calls the algorithm  $\mathcal{D}$  to process each term  $t_i$  returning a triple, i.e.,

$$(\{v_{i_1}, \cdots, v_{i_{k_i}}\}, \{(v'_{i_1}, f t_{i_1}), \cdots, (v'_{i_{k_i}}, f t_{i_{k_i}})\}, t_i[f t_{i_1} \mapsto v'_{i_1}, \cdots, f t_{i_{k_i}} \mapsto v'_{i_{k_i}}]) = \mathcal{D}[t_i].$$

The algorithm  $\mathcal{D}$  actually implements most of the algorithm in Section 4.2. It is worth noting that every time an occurrence of recursive call is found, a fresh variable is allocated for the replacement. The algorithm  $\mathcal{A}$  is correct and guaranteed to terminate with a successful hylomorphism as result.



$$\begin{aligned}
\mathcal{A}[f = \lambda v s. \text{case } t_0 \text{ of } p_1 \rightarrow t_1; \dots; p_n \rightarrow t_n] &= (f = \llbracket \phi_1 \nabla \dots \nabla \phi_n, id, \psi \rrbracket_{F,F}) \\
\text{where } F &= F_1 + \dots + F_n \\
F_i &= \mathbf{!}\mathbf{1}, && \text{if } k_i = l_i = 0 \\
&= \mathbf{!}(\Gamma(v_{i_1})) \times \dots \times \mathbf{!}(\Gamma(v_{i_{k_i}})) \times I_1 \times \dots \times I_{l_i}, && (I_1 = \dots = I_{l_i} = I), \text{ otherwise} \\
\phi_i &= \lambda (v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). t''_i \\
\psi &= \lambda v s. \text{case } t_0 \text{ of } p_1 \rightarrow (\mathbf{1}, t'_1); \dots; p_n \rightarrow (\mathbf{n}, t'_n) \\
t'_i &= (v_{i_1}, \dots, v_{i_{k_i}}, t_{i_1}, \dots, t_{i_{l_i}}) \\
\text{where} & \\
&(\{v_{i_1}, \dots, v_{i_{k_i}}\}, \{(v'_{i_1}, f t_{i_1}), \dots, (v'_{i_{l_i}}, f t_{i_{l_i}})\}, t''_i) = \mathcal{D}[t_i], \quad (i = 1, \dots, n) \\
\Gamma(v) &= \text{return } v \text{'s type} \\
\mathcal{D}[v] &= \text{if } v \text{ is a global variable then } (\{\}, \{\}, v) \text{ else } (\{v\}, \{\}, v) \\
\mathcal{D}[(t_1, \dots, t_n)] &= (s_1 \cup \dots \cup s_n, c_1 \cup \dots \cup c_n, (t'_1, \dots, t'_n)) \\
&\quad \text{where } (s_i, c_i, t'_i) = \mathcal{D}[t_i], \quad i = 1, \dots, n \\
\mathcal{D}[v t] &= \text{if } v = f \text{ then } (\{\}, \{(u, f t)\}, u) \text{ else } (s_v \cup s_t, c_v \cup c_t, t_v t_t) \\
&\quad \text{where } (s_v, c_v, t_v) = \mathcal{D}[v], (s_t, c_t, t_t) = \mathcal{D}[t] \\
&\quad \quad u \text{ is a fresh variable} \\
\mathcal{D}[C t] &= (s, c, C t') \text{ where } (s, c, t') = \mathcal{D}[t]
\end{aligned}$$

Fig. 2 Algorithm for Deriving Hyломorphisms

**Theorem 4** The algorithm  $\mathcal{A}$  is correct.

*Proof:* First,  $\llbracket \phi_1 \nabla \dots \nabla \phi_n, id, \psi \rrbracket_{F,F}$  is a correct definition of hyломorphism because (1)  $F$  is a polynomial functor; (2)  $\phi$  has the type of  $F T_o \rightarrow T_o$  and  $\psi$  has the type of  $T_i \rightarrow F T_i$ , as easily verified, where  $T_i$  and  $T_o$  denote  $f$ 's input type and output type respectively; (3)  $id$  is a natural transformation from  $F$  to  $F$ .

Next, we argue that  $f = \llbracket \phi_1 \nabla \dots \nabla \phi_n, id, \psi \rrbracket_{F,F}$  by the following calculation.

$$\begin{aligned}
&f = \llbracket \phi_1 \nabla \dots \nabla \phi_n, id, \psi \rrbracket_{F,F} \\
\equiv &\{ \text{Definition of hyломorphism} \} \\
&f = \phi_1 \nabla \dots \nabla \phi_n \circ id \circ F f \circ \psi \\
\equiv &\{ \text{Definition of } \psi \text{ and } F \} \\
&f = \phi_1 \nabla \dots \nabla \phi_n \circ (F_1 + \dots + F_n) f \circ \\
&\quad (\lambda v s. \text{case } t_0 \text{ of } p_1 \rightarrow (\mathbf{1}, t'_1); \dots; p_n \rightarrow (\mathbf{n}, t'_n)) \\
\equiv &\{ \text{Promote function into case expression} \} \\
&f = \lambda v s. \text{case } t_0 \text{ of } p_1 \rightarrow (\phi_1 \circ F_1 f) t'_1; \dots; p_n \rightarrow (\phi_n \circ F_n f) t'_n \\
\equiv &\{ \text{Definition of } t'_i \text{'s} \} \\
&f = \lambda v s. \text{case } t_0 \text{ of} \\
&\quad p_1 \rightarrow (\phi_1 \circ F_1 f) (v_{1_1}, \dots, v_{1_{k_1}}, t_{1_1}, \dots, t_{1_{l_1}}); \\
&\quad \dots; \\
&\quad p_n \rightarrow (\phi_n \circ F_n f) (v_{n_1}, \dots, v_{n_{k_n}}, t_{n_1}, \dots, t_{n_{l_n}}) \\
\equiv &\{ \text{Definition of } F_i \text{ and } \phi_i \} \\
&f = \lambda v s. \text{case } t_0 \text{ of} \\
&\quad p_1 \rightarrow (\lambda (v_{1_1}, \dots, v_{1_{k_1}}, v'_{1_1}, \dots, v'_{1_{l_1}}). t''_1) (v_{1_1}, \dots, v_{1_{k_1}}, f t_{1_1}, \dots, f t_{1_{l_1}}); \\
&\quad \dots; \\
&\quad p_n \rightarrow (\lambda (v_{n_1}, \dots, v_{n_{k_n}}, v'_{n_1}, \dots, v'_{n_{l_n}}). t''_n) (v_{n_1}, \dots, v_{n_{k_n}}, f t_{n_1}, \dots, f t_{n_{l_n}})
\end{aligned}$$

$$\begin{aligned}
&\equiv \{ \text{Simplification} \} \\
&f = \lambda v_s. \text{case } t_0 \text{ of} \\
&\quad p_1 \rightarrow t'_1[v'_1 \mapsto f t_{1_1}, \dots, v'_{1_{t_1}} \mapsto f t_{1_{t_1}}]; \\
&\quad \dots; \\
&\quad p_n \rightarrow t'_n[v'_{n_1} \mapsto f t_{n_1}, \dots, v'_{n_{t_n}} \mapsto f t_{n_{t_n}}] \\
&\equiv \{ \text{Property of } \mathcal{D} \text{ algorithm} \} \\
&f = \lambda v_s. \text{case } t_0 \text{ of } p_1 \rightarrow t_1; \dots; p_n \rightarrow t_n
\end{aligned}$$

□

#### 4.4 Examples

To see the algorithm  $\mathcal{A}$  in action, consider some examples defined in Section 3.

We begin by considering a quite simple definition of *sum*. In this case, we know that  $t_1 = 0$  and  $t_2 = \text{plus}(a, \text{sum } as)$ . Applying  $\mathcal{D}$  to  $t_1$  and  $t_2$  gives

$$\begin{aligned}
\mathcal{D}[0] &= (\{\}, \{\}, 0) \\
\mathcal{D}[\text{plus}(a, \text{sum } as)] &= (\{a\}, \{(v'_1, \text{sum } as)\}, \text{plus}(a, v'_1)).
\end{aligned}$$

It follows from  $\mathcal{A}$  that

$$\begin{aligned}
t'_1 &= (), & \phi_1 &= \lambda(). 0 = 0 \\
t'_2 &= (a, as), & \phi_2 &= \lambda(a, v'_1). \text{plus}(a, v'_1)
\end{aligned}$$

and

$$\begin{aligned}
\psi &= \lambda xs. \text{case } xs \text{ of } Nil \rightarrow (1, ()); \text{Cons}(a, as) \rightarrow (2, (a, as)) = \text{out}_F \\
F &= !\mathbf{1} + !Int \times I
\end{aligned}$$

Therefore we derive the following hylomorphism for *sum*:

$$\text{sum} = \llbracket 0 \nabla \lambda(a, v'_1). \text{plus}(a, v'_1), id, \text{out}_F \rrbracket_{F,F}.$$

The above hylomorphism is also catamorphism  $(\llbracket 0 \nabla \lambda(a, v'_1). \text{plus}(a, v'_1) \rrbracket)_F$ .

Our second example is to deal with the recursive definition of *foldr1*  $\oplus$ . This also appeared in [GLJ93] for the illustration of the limitation of their shortcut deforestation algorithm because *foldr1*  $\oplus$  cannot be specified as a catamorphism (since it does not treat all *Cons* cells identically). With the algorithm  $\mathcal{A}$ , we can get the following hylomorphism.

$$\begin{aligned}
\text{foldr1 } \oplus &= \llbracket \phi, id, \psi \rrbracket_{F,F} \\
&\text{where } F = !\mathbf{1} + !B + !B \times I \\
&\phi = \phi_1 \nabla \phi_2 \nabla \phi_3 \\
&\quad \text{where } \phi_1 = \text{error} \\
&\quad \phi_2 = \lambda a. a = id \\
&\quad \phi_3 = \lambda(a, v'_1). (a \oplus v'_1) = \oplus \\
\psi &= \lambda xs. \text{case } xs \text{ of } Nil \rightarrow (1, ()); \\
&\quad \text{Cons}(a, Nil) \rightarrow (2, (a)); \\
&\quad \text{Cons}(a, as) \rightarrow (3, (a, as))
\end{aligned}$$

Section 6 will show how it helps removing more intermediate data structures than [GLJ93].

Next we'd like to show that our derivation algorithm does not restrict to the recursive definitions inducting over a single parameter. Consider the recursive definition of *zip* which inducts over multiple parameters. Applying algorithm  $\mathcal{A}$  will give the following hylomorphism.

$$\begin{aligned} \text{zip} &= \llbracket Nil \nabla Nil \nabla \lambda(a, b, p). \text{Cons}((a, b), p), id, \psi \rrbracket_{F, F} \\ &\text{where } F = !\mathbf{1} + !\mathbf{1} + !A \times !B \times I \\ &\quad \psi = \lambda(xs, ys). \text{ case } (xs, ys) \text{ of} \\ &\quad \quad (Nil, \_) \rightarrow (1, ()); \\ &\quad \quad (\text{Cons}(a, as), Nil) \rightarrow (2, ()); \\ &\quad \quad (\text{Cons}(a, as), \text{Cons}(b, bs)) \rightarrow (3, (a, b, (as, bs))) \end{aligned}$$

The advantage of this transformation is that *zip* now can be fused with other functions by the Hylo Fusion Theorem. Compared with Fegaras's approach [FSZ94] where some new fusion theorems were intensively developed, our algorithm makes it unnecessary to obtain the same effect. More detail discussion can be found in [HIT95].

Finally, we give some other examples.

$$\text{map } g = \llbracket Nil \nabla \lambda(a, v'_1). \text{Cons}(g a, v'_1), id, out_{L_A} \rrbracket_{L_A, L_A}$$

$$\begin{aligned} \text{upto} &= \llbracket in_{L_{Int}}, id, \psi \rrbracket_{L_{Int}, L_{Int}} \quad \text{where} \\ \psi &= \lambda(b, e). \text{ case } b < e \text{ of} \\ &\quad \text{True} \rightarrow (1, ()); \\ &\quad \text{False} \rightarrow (2, (b, (\text{plus}(b, 1), e))) \end{aligned}$$

## 5 Restructuring Hylomorphisms

Once a hylomorphism is got, it can be fused with other functions. But not all hylomorphisms have good structures for program fusion to be effectively applied. In this section, we show how to structure the given hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G, F}$  by arranging  $\phi$ ,  $\eta$  and  $\psi$  well inside it, so that it could be fused with other functions effectively by the two fusion theorems, namely the Hylo Fusion and the Acid Rain Theorems.

### 5.1 Suitable Hylomorphisms for Hylo Fusion Theorem

The effective use of the Hylo Fusion Theorem requires that  $\phi$  ( $\psi$ ) in a hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G, F}$  contain as much computation as possible for the Left (Right) Fusion Law. As an example, consider the following program *foo*, where  $N$  represents the type of natural numbers with two constructors *Zero* and *Succ*.

$$\begin{aligned} \text{foo} &: List\ N \rightarrow List\ N \\ \text{foo} &= h \circ \llbracket in_{L_N}, id, (id + Succ \times id) \circ out_{L_N} \rrbracket_{L_N, L_N} \\ h &= \lambda xs. \text{ case } xs \text{ of} \\ &\quad Nil \rightarrow Nil; \\ &\quad \text{Cons}(Zero, Nil) \rightarrow Nil; \\ &\quad \text{Cons}(Zero, \text{Cons}(x', xs')) \rightarrow \text{Cons}(Succ\ x', h\ xs'); \\ &\quad \text{Cons}(Succ\ x', xs') \rightarrow \text{Cons}(Succ\ x', h\ xs') \end{aligned}$$

To fuse  $foo$  by the Left Fusion Law, we have to derive  $\phi'$  from the equation  $h \circ in_{L_N} = \phi' \circ L_N h$ . But such  $\phi'$  cannot be derived because of the lack of information in  $in_{L_N}$ . To solve this problem, we shift some computations to  $in_{L_N}$  and get the following program:

$$foo = h \circ \llbracket in_{L_N} \circ (id + Succ \times id), id, out_{L_N} \rrbracket_{L_N, L_N}.$$

Because of the knowledge that the value of  $Succ x$  cannot be  $Zero$  and thus the second and third branches of the case expression in  $h$  cannot be taken, we can derive that  $\phi' = Nil \nabla (Cons \circ (Succ \times id))$  and have  $foo = \llbracket \phi', id, out_{L_N} \rrbracket_{L_N, L_N}$ .

Similar cases might happen to the Right Fusion Law.

## 5.2 Suitable Hylomorphisms for Acid Rain Theorem

The Acid Rain Theorem expects that  $\phi$  and  $\psi$  in the hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G, F} : A \rightarrow B$  to be described as  $\tau in_{F_B}$  and  $\sigma out_{F_A}$  respectively. Here,  $\tau$  and  $\sigma$  are polymorphic functions and  $F_A$  and  $F_B$  are functors defining types  $A$  and  $B$  respectively. Our Laws for deriving such  $\tau$  and  $\sigma$  are as follows.

**Theorem 5 (Deriving Polymorphic Function)** Under the above conditions,  $\tau$  and  $\sigma$  are defined by the following two laws.

$$\frac{\forall \alpha. \ (\llbracket \alpha \rrbracket_{F_B} \circ \phi = \phi' \circ G \llbracket \alpha \rrbracket_{F_B})}{\tau = \lambda \alpha. \ \phi'}$$

$$\frac{\forall \beta. \ \psi \circ \llbracket \beta \rrbracket_{F_A} = F \llbracket \beta \rrbracket_{F_A} \circ \psi'}{\sigma = \lambda \beta. \ \psi'}$$

*Proof sketch:* We only show the correctness of the above law for defining  $\tau$ .

(1)  $\tau$  has the type of  $\forall A. (F_B A \rightarrow A) \rightarrow G A \rightarrow A$  as required because  $\tau$  is defined for all  $\alpha : F_B A \rightarrow A$  with any  $A$ ;

(2) We prove that  $\phi = \tau in_{F_B}$  by the following calculation.

$$\begin{aligned} & \phi = \tau in_{F_B} \\ \equiv & \quad \{ \text{Definition of } \tau \} \\ & \phi = \phi'[\alpha \mapsto in_{F_B}] \\ \equiv & \quad \{ \text{Since } (\llbracket in_{F_B} \rrbracket_{F_B} = id \text{ and } G id = id) \} \\ & (\llbracket in_{F_B} \rrbracket_{F_B})_{F_B} \circ \phi = \phi'[\alpha \mapsto in_{F_B}] \circ G (\llbracket in_{F_B} \rrbracket_{F_B})_{F_B} \\ \Leftarrow & \quad \{ \text{Assumption} \} \\ & \forall \alpha. \ (\llbracket \alpha \rrbracket_{F_B} \circ \phi = \phi' \circ G \llbracket \alpha \rrbracket_{F_B}) \quad \square \end{aligned}$$

Because in using the Acid Rain Theorem we have to derive  $\phi'$  ( $\psi'$ ) from  $\phi$  ( $\psi$ ) for any  $\alpha$  ( $\beta$ ), it is expected that  $\phi$  ( $\psi$ ) is simple (contains few computations).

## 5.3 Algorithm for Structuring Hylomorphisms

Generally, the behavior of a hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G, F}$  could be understood as follows:  $\psi$  generates a recursive structure,  $\eta$  operates on the elements of the structure, and  $\phi$  manipulates on the recursive structure. It is possible for  $\phi$  and  $\psi$

to have the computation that  $\eta$  can do. They are said to have the *least computation* if they do not have computation that  $\eta$  can do.

A hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket$  is said to be *structural* if  $\phi$  and  $\psi$  contain the least computation. Structural hylomorphisms are fit for the two fusion theorems. For the Acid Rain Theorem, it is fitful since  $\phi$  and  $\psi$  are simple. For the Hylo Fusion Theorem, since  $\phi$  ( $\psi$ ) contain the least computation, it implies that  $\eta \circ \psi$  ( $\phi \circ \eta$ ) contains the most computation and so  $\llbracket \phi, id, \eta \circ \psi \rrbracket$  ( $\llbracket \phi \circ \eta, id, \psi \rrbracket$ ) is suitable for the Right (Left) Fusion Law. In other words, once a structural hylomorphism is got, we can “shift” the natural transformation ( $\eta$ ) freely inside the hylomorphism according to which Fusion Law is to be applied.

Our algorithm for structuring the given hylomorphism  $\llbracket \phi, \eta, \psi \rrbracket_{G,F}$  is to shift computations from  $\phi$  and  $\psi$  into  $\eta$  by factorizing  $\phi$  to  $\phi' \circ \eta_\phi$  and  $\psi$  to  $\eta_\psi \circ \psi'$  so that  $\phi'$  and  $\psi'$  contain the least computation, resulting in a structural hylomorphism  $\llbracket \phi', \eta_\phi \circ \eta \circ \eta_\psi, \psi' \rrbracket_{G',F'}$ . In the following, we give the algorithm for factorizing  $\phi$  to  $\phi' \circ \eta_\phi$ , while omitting the dual discussion on  $\psi$ .

Let  $\phi : G A \rightarrow A$  be given as:

$$\phi = \phi_1 \nabla \cdots \nabla \phi_n$$

where  $G = G_1 + \cdots + G_n$  and  $\phi_i : G_i A \rightarrow A$ . A typical  $\phi_i$ , as in algorithm  $\mathcal{A}$ , is defined by:

$$\phi_i = \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). t_i$$

The variables with type  $A$  are explicitly indicated with  $'$  attached and called *recursive variables*.

In order to capture the computations in  $\phi_i$  which can be done by a natural transformation, we define *maximal non-recursive subterms* as follows.

**Definition 10 (Maximal Non-Recursive Subterm)** A term  $t_{i_j}$  is said to be a *non-recursive subterm* of  $t_i$  if (1)  $t_{i_j}$  is a subterm of  $t_i$ ; (2)  $t_{i_j}$  does not include recursive variables. A non-recursive subterm is said to be *maximal* if it is not a subterm of other non-recursive subterms.  $\square$

The essence of our algorithm is to factorize  $\phi_i$  into  $\phi'_i \circ \eta_{\phi_i}$  so that all the maximal non-recursive subterms in  $t_i$  are shifted into  $\eta_{\phi_i}$ . Informally, the algorithm for factorizing  $\phi_i$  is as follows.

1. Find all the maximal non-recursive subterms in  $t_i$ , say  $t_{i_1}, \dots, t_{i_{m_i}}$ .
2. Let  $t'_i$  be the term from  $t_i$  with each maximal non-recursive subterm  $t_{i_j}$  be replaced by a new variable  $u_{i_j}$ , i.e.,  $t'_i = t_i[t_{i_1} \mapsto u_{i_1}, \dots, t_{i_{m_i}} \mapsto u_{i_{m_i}}]$ .
3. Factorize  $\phi_i$  by extracting all the maximal non-recursive subterms out of  $t_i$  as follows.

$$\begin{aligned} \phi_i &= \phi'_i \circ \eta_{\phi_i} \\ \text{where } \phi'_i &= \lambda(u_{i_1}, \dots, u_{i_{m_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). t'_i \\ \eta_{\phi_i} &= \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). (t_{i_1}, \dots, t_{i_{m_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}) \end{aligned}$$

4. Factorize  $\phi$  by grouping the result of  $\phi_i$ , i.e.,  $\phi = (\phi'_1 \nabla \cdots \nabla \phi'_n) \circ (\eta_{\phi_1} + \cdots + \eta_{\phi_n})$ .

$$\begin{aligned}
\mathcal{S}[[\phi_1 \nabla \cdots \nabla \phi_n, \eta, \psi]_{G,F}] &= [[\phi'_1 \nabla \cdots \nabla \phi'_n, (\eta_{\phi_1} + \cdots + \eta_{\phi_n}) \circ \eta, \psi]_{G',F}] \\
\text{where } \phi'_i &= \lambda(u_{i_1}, \dots, u_{i_m}, v'_{i_1}, \dots, v'_{i_{l_i}}). t'_i \\
\eta_{\phi_i} &= \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). (t_{i_1}, \dots, t_{i_{m_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}) \\
G' &= G'_1 + \cdots + G'_n \\
G_i &= !\mathbf{1}, && \text{if } m_i = l_i = 0 \\
&= !(\Gamma(u_{i_1})) \times \cdots \times !(\Gamma(u_{i_{m_i}})) \times I_1 \times \cdots \times I_{l_i}, && (I_1 = \cdots = I_{l_i} = I), \text{ otherwise} \\
\text{where } \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). t_i &= \phi_i, && (\text{assume } v'_{i_1}, \dots, v'_{i_{l_i}} \text{ are recursive parameters}) \\
&(\{(u_{i_1}, t_{i_1}), \dots, (u_{i_{m_i}}, t_{i_{m_i}})\}, t'_i) = \mathcal{E}[t_i] \{v'_{i_1}, \dots, v'_{i_{l_i}}\} \\
\mathcal{E}[v] s_r &= \text{if } v \in s_r \text{ then } (\{\}, v) \text{ else } (\{(u, v)\}, u) \\
&\quad \text{where } u \text{ is a fresh variable} \\
\mathcal{E}[(t_1, \dots, t_n)] s_r &= \text{if } \forall i, \text{Var}_{s_r}(t_i) \text{ then } (\{(u, (t_1, \dots, t_n))\}, u) \text{ else } (w_1 \cup \cdots \cup w_n, (t'_1, \dots, t'_n)) \\
&\quad \text{where } (w_i, t'_i) = \mathcal{E}[t_i] s_r \ (i = 1, \dots, n), \ u \text{ is a fresh variable} \\
\mathcal{E}[C t] s_r &= \text{if } \text{Var}_{s_r}(t) \text{ then } (\{(u, C t)\}, u) \text{ else } (w, C t') \\
&\quad \text{where } (w, t') = \mathcal{E}[t] s_r, \ u \text{ is a fresh variable} \\
\mathcal{E}[v t] s_r &= \text{if } \text{Var}_{s_r}(t'_v) \wedge \text{Var}_{s_r}(t'_t) \text{ then } (\{(u, v t)\}, u) \text{ else } (w_v \cup w_t, t'_v t'_t) \\
&\quad \text{where } (w_v, t'_v) = \mathcal{E}[v] s_r, \ (w_t, t'_t) = \mathcal{E}[t] s_r, \ u \text{ is a fresh variable} \\
\text{Var}_{s_r}(t) &= t \text{ is a variable } \wedge t \notin s_r
\end{aligned}$$

**Fig. 3** Algorithm for Structuring Hylomorphisms

The above algorithm is summarized in Fig. 3. The main transformation is  $\mathcal{S}$  which in turn calls transformation  $\mathcal{E}$  to process every term  $t_i$  for factorizing  $\phi_i$ . Similar to the algorithm in Fig. 2, a fresh variable is allocated every time a non-recursive subterm is found. It will be discarded when the corresponding non-recursive subterm turns out to be non-maximal by the predicate  $\text{Var}$ . The correctness of the algorithm  $\mathcal{S}$  is omitted, but it should be noted that  $\phi'_1 \nabla \cdots \nabla \phi'_n : G' A \rightarrow A$  and  $\eta_{\phi_1} + \cdots + \eta_{\phi_n}$  is natural transformation from  $G$  to  $G'$ .

**Theorem 6** The  $\eta_{\phi_1} + \cdots + \eta_{\phi_n}$ , derived in the algorithm  $\mathcal{S}$ , is natural transformation from  $G$  to  $G'$ , i.e.,

$$\eta_{\phi_1} + \cdots + \eta_{\phi_n} : G \rightarrow G'.$$

*Proof:* We prove it by the following calculation.

$$\begin{aligned}
&\eta_{\phi_1} + \cdots + \eta_{\phi_n} : G \rightarrow G' \\
\equiv &\{ \text{Definition of natural transformation} \} \\
&\forall f. (\eta_{\phi_1} + \cdots + \eta_{\phi_n}) \circ G f = G' f \circ (\eta_{\phi_1} + \cdots + \eta_{\phi_n}) \\
\equiv &\{ \text{Definitions of } G \text{ and } G' \} \\
&\forall f. \eta_{\phi_1} \circ G_1 f + \cdots + \eta_{\phi_n} \circ G_n f = G'_1 f \circ \eta_{\phi_1} + \cdots + G'_n f \circ \eta_{\phi_n} \\
\Leftarrow &\{ \text{trivial} \} \\
&\forall f. \eta_{\phi_i} \circ G_i f = G'_i f \circ \eta_{\phi_i}, \ (i = 1, \dots, n) \\
\equiv &\{ \text{Definitions of } \eta_{\phi_i}, G_i \text{ and } G'_i \} \\
&\forall f. \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). (t_{i_1}, \dots, t_{i_{m_i}}, f v'_{i_1}, \dots, f v'_{i_{l_i}}) \\
&\quad = \lambda(v_{i_1}, \dots, v_{i_{k_i}}, v'_{i_1}, \dots, v'_{i_{l_i}}). (t_{i_1}, \dots, t_{i_{m_i}}, f v'_{i_1}, \dots, f v'_{i_{l_i}}), \ (i = 1, \dots, n) \\
\equiv &\{ \text{obvious} \} \\
&\text{True}
\end{aligned}$$

□

## 5.4 Examples

Let's structure the following hylomorphism obtained in Section 4.4 where  $g : A \rightarrow B$ .

$$\text{map } g = \llbracket Nil \nabla \lambda(a, v'_1). \underline{Cons}(g a, v'_1), id, out_{LA} \rrbracket_{LA, LA}$$

In this example,  $\phi_1 = \lambda(). Nil$ , and  $\phi_2 = \lambda(a, v'_1). \underline{Cons}(g a, v'_1)$ . Here there is only one maximal non-recursive subterm as underlined. Our algorithm will move out the computation  $g a$  out of  $\phi_2$  as

$$\phi_2 = \phi'_2 \circ \eta_{\phi_2}, \text{ where } \phi'_2 = \lambda(u_{21}, v'_1). \underline{Cons}(u_{21}, v'_1), \eta_{\phi_2} = \lambda(a, v'_1).(g a, v'_1)$$

and finally give the following structural hylomorphism:

$$\text{map } g = \llbracket Nil \nabla \lambda(u_{21}, v'_1). \underline{Cons}(u_{21}, v'_1), id + \lambda(a, v'_1).(g a, v'_1), out_{LA} \rrbracket_{LB, LA}.$$

Moreover, with Theorem 5, we can get  $\phi' = \alpha$  and  $\tau = id$ . So the above hylomorphism becomes  $\llbracket in_{LB}, id + \lambda(a, v'_1).(g a, v'_1), out_{LA} \rrbracket_{LB, LA}$ , a structural hylomorphism for the two fusion theorems.

## 6 Related Work and Discussions

It has been argued that programming with the use of generic control structures which capture patterns of recursions in a uniform way is very significant in program transformation and optimization[GLJ93, MFP91, SF93, TM95]. Our work is much related to these studies. In particular, our work was greatly motivated by Sheard and Fegaras's work[SF93] and Takano and Meijer's[TM95].

Both of them, essentially, work with a *language without general recursion but containing hylomorphisms as basic components*, advocating *structural functional programming*. Sheard and Fegaras implemented a fusion algorithm called *normalization algorithm*<sup>2</sup> based on the similar theorem like the Hylo Fusion Theorem. Takano and Meijer generalized Gill, Launchbury and Peyton Jones's one-step fusion algorithm relying on functions being written in a highly-stylized *build-cata* forms[GLJ93] (i.e., catamorphisms with data constructors being parameterized), and implemented another one-step fusion algorithm based on the Acid Rain Theorem. All have made it clear that the *fusion process is especially successful if the recursive definitions are expressed in terms of hylomorphisms*.

However, as argued by Launchbury and Sheard[LS95], although structural functional programming contributes much to program transformation and optimization, it is unrealistic in real functional programming. It is impractical to force programmers to define their recursive definitions only in terms of the specific forms like hylomorphisms, in which a lot of abstract categorical concepts are embedded.

To remedy this situation, Launchbury and Sheard[LS95] gave an algorithm to turn recursive definitions into build-cata forms. The major problem still left is

<sup>2</sup> Sheard and Fegaras's normalization algorithm first only worked with the language containing folds (i.e., catamorphisms) as basic components. It has been extended to work with languages containing so-called homomorphisms (i.e., hylomorphisms) in [KL94].

that many programs cannot be fused because some recursive definitions cannot be specified in build-cata forms. As an example, consider the program

$$\text{foldr1 } \oplus \circ \text{map } g$$

which accepts a list  $\text{Cons}(x_1, \dots, \text{Cons}(x_{n-1}, \text{Cons}(x_n, \text{Nil})) \dots)$  and returns  $g x_1 \oplus (\dots (g x_{n-1} \oplus g x_n) \dots)$ . Since  $\text{foldr1 } \oplus$  is not a catamorphism (Section 4.4), the algorithm in [LS95] will fail and leave the intermediate data structure produced by  $\text{map } g$  remained. Our algorithm can solve this problem. With the results in Sections 4.4 and 5.4, we have

$$\llbracket \text{error } \nabla \text{id } \nabla \oplus, \text{id}, \psi \rrbracket_{F,F} \circ \llbracket \text{in}_{L_B}, \text{id} + \lambda(a, v'_1).(g a, v'_1), \text{out}_{L_A} \rrbracket_{L_B, L_A},$$

and we can derive a polymorphic function

$$\sigma = \lambda\beta. (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times \beta^{-1} \circ \iota_2)) \circ \text{dist}) \circ L_A \beta \circ \beta$$

from  $\psi$  such that  $\psi = \sigma \text{out}_{L_B}$  according to Theorem 5, where  $\text{dist}$  is a natural transformation defined as follows.

$$\begin{aligned} \text{dist} & : X \times (Y + Z) \rightarrow X \times Y + X \times Z \\ \text{dist}(x, (1, y)) & = (1, (x, y)) \\ \text{dist}(x, (2, z)) & = (2, (x, y)) \end{aligned}$$

In the following, we demonstrate how the Ana-Hylo Fusion Law is applied to eliminate the intermediate data structure in the program  $\text{foldr1 } \oplus \circ \text{map } g$ .

$$\begin{aligned} & \text{foldr1 } \oplus \circ \text{map } g \\ = & \{ \text{Replace } \text{foldr1 } \oplus \text{ and } \text{map } g \text{ with their hylomorphisms } \} \\ & \llbracket \text{error } \nabla \text{id } \nabla \oplus, \text{id}, \sigma \text{out}_{L_B} \rrbracket_{F,F} \circ \llbracket \text{in}_{L_B}, \text{id} + g \times \text{id}, \text{out}_{L_A} \rrbracket_{L_B, L_A} \\ = & \{ \text{Acid Rain Theorem (Hylo-Ana Fusion Law)} \} \\ & \llbracket \text{error } \nabla \text{id } \nabla \oplus, \text{id}, \sigma((\text{id} + g \times \text{id}) \circ \text{out}_{L_A}) \rrbracket_{F,F} \end{aligned}$$

Now we focus on the transformation of the third part in the above hylomorphism.

$$\begin{aligned} & \sigma((\text{id} + g \times \text{id}) \circ \text{out}_{L_A}) \\ = & \{ \text{Definition of } \sigma, \text{ let } h = \text{id} + g \times \text{id} \} \\ & (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times (h \circ \text{out}_{L_A})^{-1} \circ \iota_2)) \circ \text{dist}) \circ L_A (h \circ \text{out}_{L_A}) \circ h \circ \text{out}_{L_A} \\ = & \{ \text{Definition of } L_A, \text{ and } h. \} \\ & (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times (h \circ \text{out}_{L_A})^{-1} \circ \iota_2)) \circ \text{dist}) \\ & \circ (\text{id} + (g \times (h \circ \text{out}_{L_A}))) \circ \text{out}_{L_A} \\ = & \{ \text{Definition of } L_A \} \\ & (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times (h \circ \text{out}_{L_A})^{-1} \circ \iota_2)) \circ \text{dist}) \\ & \circ (\text{id} + g \times h) \circ L_A \text{out}_{L_A} \circ \text{out}_{L_A} \\ = & \{ \text{Definition of } h \} \\ & (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times (h \circ \text{out}_{L_A})^{-1} \circ \iota_2)) \circ \text{dist}) \\ & \circ (\text{id} + g \times (\text{id} + g \times \text{id})) \circ L_A \text{out}_{L_A} \circ \text{out}_{L_A} \\ = & \{ \text{Move } \text{dist} \text{ backwards} \} \\ & (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (\text{id} \times (h \circ \text{out}_{L_A})^{-1} \circ \iota_2))) \\ & \circ (\text{id} + \text{dist} \circ (g \times (\text{id} + g \times \text{id}))) \circ L_A \text{out}_{L_A} \circ \text{out}_{L_A} \end{aligned}$$



$$\begin{aligned}
&= \{ \text{Transformation property of } dist \} \\
&\quad (\iota_1 \nabla (\iota_2 \circ \pi_1 \nabla \iota_3 \circ (id \times (h \circ out_{L_A})^{-1} \circ \iota_2))) \\
&\quad \circ (id + (g \times id + g \times (g \times id)) \circ dist) \circ L_A out_{L_A} \circ out_{L_A} \\
&= \{ \text{Since } (f \nabla g) \circ (p + q) = f \circ p \nabla g \circ q, out_{L_A}^{-1} = in_{L_A} \} \\
&\quad (\iota_1 \nabla (\iota_2 \circ g \circ \pi_1 \nabla \iota_3 \circ (g \times (in_{L_A} \circ h^{-1} \circ \iota_2 \circ (g \times id)))) \circ dist) \circ L_A out_{L_A} \circ out_{L_A} \\
&= \{ \text{Since } h^{-1} \circ \iota_2 = \iota_2 \circ (g \times id)^{-1} \} \\
&\quad (\iota_1 \nabla (\iota_2 \circ g \circ \pi_1 \nabla \iota_3 \circ (g \times in_{L_A} \circ \iota_2))) \circ dist) \circ L_A out_{L_A} \circ out_{L_A}
\end{aligned}$$

It then derived the following hylomorphism:

$$\llbracket error \nabla id \nabla \oplus, id, \nabla \iota_3 \circ (g \times in_{L_A} \circ \iota_2) \rrbracket \circ dist) \circ L_A out_{L_A} \circ out_{L_A} \rrbracket_{F,F}.$$

Inlining the above derived hylomorphism, named *prg*, would give the following familiar program:

$$\begin{aligned}
prg &= \lambda xs. \text{ case } xs \text{ of} \\
&\quad Nil \rightarrow error; \\
&\quad Cons(a, Nil) \rightarrow g a \\
&\quad Cons(a, as) \rightarrow g a \oplus prg as
\end{aligned}$$

where the intermediate data structure produced by *map g* no longer exists. Compared with Launchbury and Sheard's algorithm, ours is more general and powerful.

Our work is also related to the discussion on the fusion of recursions inducting over multiple data structures. Rather than introducing new fusion theorems as in [FSZ94], we can handle it by turning them into hylomorphisms.

## 7 Conclusion

We have successfully given the algorithm to turn recursive definitions to structural hylomorphisms in order to enable program fusion. Our algorithm is a *two-stage abstraction*, abstracting recursive function calls to derive hylomorphisms and abstracting maximal non-recursive subterms to structure hylomorphisms. A prototype of our algorithm has been implemented and tested extensively, showing its promise to be embedded in a real system. Finally, our algorithm gives the evidence that hylomorphisms can specify all recursions of interests as claimed in [BdM94].

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